Mixed Hodge Complexes

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Fix R to be $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ throughout all sections. The field of fractions $R \otimes \mathbb{Q}$ will be $\mathbb{Q}, \mathbb{Q}$ or $\mathbb{R}$, respectively.

Filtrations F will always be decreasing, and filtrations W will always be increasing.

This talk is based on Section 3.3 of [Hodge]. The goal is to define (cohomological) mixed Hodge complexes and give constructions

$$\text{cohomological MHC} \implies \text{MHC} \implies \text{MHS}$$

with the goal of showing every mixed Hodge complex has a mixed Hodge structure on its cohomology.

1 Derived categories of filtered objects

Let $\mathcal{A}$ be an abelian category. From this one defines the following categories.

- $F\mathcal{A}$ the category of filtered objects of $\mathcal{A}$ with finite filtrations,
- $FW\mathcal{A}$ the category of bi-filtered objects of $\mathcal{A}$ with finite filtrations,
- $C^+F\mathcal{A}$ the category of complexes of $F\mathcal{A}$ bounded from below,
- $C^+FW\mathcal{A}$ the category of complexes of $FW\mathcal{A}$ bounded from below
- $K^+F\mathcal{A}$ be the homotopy category of filtered complexes bounded from below
  (A *homotopy* between morphisms of complexes of filtered objects is simply a homotopy between the morphisms of underlying complexes, compatible with the filtrations.)
- $K^+FW\mathcal{A}$ be the homotopy category of bi-filtered complexes bounded from below
  (Again a *homotopy* should be compatible with both filtrations.)
- $D^+F\mathcal{A}$ be obtained from $K^+F\mathcal{A}$ by inverting *filtered quasi-isomorphisms*
- $D^+FW\mathcal{A}$ be obtained from $K^+FW\mathcal{A}$ by inverting *bi-filtered quasi-isomorphisms*

To understand the last two categories, we must define what are '(bi-)filtered quasi-isomorphisms'.

Recall that a filtration on a complex K is *biregular* if the filtration on K^n is finite for each n.

Definition 1. A morphism $f : (K, F) \to (K', F')$ of complexes with biregular filtrations is a *filtered quasi-isomorphism* if $\mathrm{Gr}_F(f)$ is a quasi-isomorphism.

A morphism $f : (K, W, F) \to (K', W', F')$ of complexes with biregular filtrations is a *bi-filtered quasi-isomorphism* if $\mathrm{Gr}_F \mathrm{Gr}^W(f)$ is a quasi-isomorphism.

Remark 2. The above definition used the following. If a complex K has two filtrations, F and W, then F induces by restriction a new filtration F on the terms $W_k K$, induces a quotient filtration on the graded pieces $\mathrm{Gr}_k^W(K)$. This yields graded complexes $\mathrm{Gr}_F(K)$, $\mathrm{Gr}^W(K)$ and $\mathrm{Gr}_F \mathrm{Gr}^W(K)$.

Remark 3. Note that the cone $C(f)$ of a morphism of (bi-)filtered complexes is also naturally (bi-)filtered. This naturally gives the notion of an exact triangles in $K^+F\mathcal{A}$ and $D^+F\mathcal{A}$.

2 Derived functors on filtered objects

Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, and suppose $\mathcal{A}$ has enough injectives.

Lemma 4. *Let (A, F) be a filtered object in $F\mathcal{A}$. Since T is left exact, the subobjects $TF^p(A)$ define a filtration TF of TA. If $\mathrm{Gr}_F(A)$ is T-acyclic, then $\mathrm{Gr}_{TF}(TA) \cong T(\mathrm{Gr}_F A)$.*

Proof. If $\mathrm{Gr}_F(A)$ is T-acyclic, the subobjects F^pA are T-acyclic as successive extensions of T-acyclic objects. Hence the image under T of the exact sequence

$$0 \to F^{p+1}(A) \to F^p(A) \to \mathrm{Gr}_F^p(A) \to 0$$

is exact. □

Definition 5. Let (K, F) be a filtered complex with biregular filtrations. A *filtered T-acyclic resolution* of K is a filtered quasi-isomorphism $f : (K, F) \to (K', F')$, with F' also biregular, such that $\mathrm{Gr}_F^p(K'^n)$ is T-acylic for all n and p.

Proposition 6. *Suppose we are given functorially for each filtered complex (K, F) a filtered T-acyclic resolution $i : (K, F) \to (K', F')$. Define $T' : C^+F\mathcal{A} \to D^+F\mathcal{B}$ by $T'(K, F) = (TK', TF')$. Then T' sends filtered quasi-isomorphisms to isomorphisms in $D^+F\mathcal{B}$, and hence T' factors through a derived functor*

$$RT : D^+F\mathcal{A} \to D^+F\mathcal{B}$$

such that $RT(K, F) = (TK', TF')$. Moreover, $\mathrm{Gr}_F RT(K) \cong RT(\mathrm{Gr}_F K) \cong \mathrm{Gr}_{TF'}(TK')$.

Example 7. Let K be a complex with the stupid filtration $F^p(K) = \sigma_{\geq p}(K)$. One can find a T-acyclic resolution using the Cartan–Eilenberg resolution $K \to \mathrm{Tot}(I^{\bullet,\bullet})$, where $I^{n,\bullet}$ are T-acyclic resolutions of K. There is a suitable filtration F' on $K' = \mathrm{Tot}(I^{\bullet,\bullet})$ such that the morphism $(K, F) \to (K', F')$ is a filtered quasi-isomorphism.

3 Derived functors on bi-filtered objects

Definition 8. Let (K, W, F) be a bi-filtered complex with biregular filtrations. A *bi-filtered T-acyclic resolution* of K is a bi-filtered quasi-isomorphism $f : (K, W, F) \to (K', W', F')$, with F' and W' also biregular, such that $\mathrm{Gr}_F^p \mathrm{Gr}_q^W(K'^n)$ are T-acylic for all n, p and q.

Proposition 9. *Suppose we are given functorially for every bi-filtered complex (K, W, F) a bi-filtered T-acyclic resolution $i : (K, W, F) \to (K', W', F')$. Define $T' : C^+FW\mathcal{A} \to D^+FW\mathcal{B}$ by $T'(K, W, F) = (TK', TF', TW')$. Then T' sends bi-filtered quasi-isomorphisms to isomorphisms in $D^+FW\mathcal{B}$, and hence T' factors through a derived functor*

$$RT : D^+FW\mathcal{A} \to D^+FW\mathcal{B}.$$

Moreover, $\mathrm{Gr}_F \mathrm{Gr}^W (RT(K)) \cong RT(\mathrm{Gr}_F \mathrm{Gr}^W K)$.

4 Hodge complexes

Definition 10. An *R-Hodge complex* K of weight n consists of

- a complex K_R of R-modules, such that all $H^i(K_R)$ are finitely generated R-modules,
- a filtered complex $(K_\mathbb{C}, F)$ of $\mathbb{C}$-vector spaces,
- an isomorphism $\alpha : K_R \otimes \mathbb{C} \xrightarrow{\sim} K_\mathbb{C}$ in $D^+(\mathbb{C})$,

such that

(HC 1) the differentials d^i are strict, that is, $d^i(F^p(K_R^i)) = d^i(K_R^i) \cap F^p(K_R^{i+1})$,

(HC 2) the induced filtration F on $H^i(K_\mathbb{C}) \cong H^i(K_R) \otimes \mathbb{C}$ defines an R-Hodge structure of weight $n + i$ on $H^i(K_R)$.

Definition 11. Let X be a topological space. An *R-cohomological Hodge complex* K of weight n on X consists of

- a complex K_R of sheaves of R-modules on X,
- a filtered complex $(K_\mathbb{C}, F)$ of sheaves of $\mathbb{C}$-vector spaces on X,
- an isomorphism $\alpha : K_R \otimes \mathbb{C} \xrightarrow{\sim} K_\mathbb{C}$ in $D^+(X, \mathbb{C})$,

such that

(CHC) the triple $(R\Gamma(X, K_R), R\Gamma(X, K_\mathbb{C}, F), R\Gamma(\alpha))$ is an R-Hodge complex of weight n.

Remark 12. If (K, F) is a (cohomological) Hodge complex of weight n, then $(K[m], F[p])$ is a (cohomological) Hodge complex of weight $n + m - 2p$.

Remark 13. The Hodge decomposition theorem may be stated as follows. Let X be a compact complex algebraic manifold. Let $K_\mathbb{Z}$ be the constant sheaf $\mathbb{Z}$ on X concentrated in degree 0. Let $K_\mathbb{C} = \Omega_X^\bullet$ be the analytic de Rham complex with its stupid filtration by subcomplexes

$$F^p\Omega_X^\bullet = \left[0 \to \cdots \to 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots \to \Omega_X^n \to 0\right].$$

Let $\alpha : K_\mathbb{Z} \otimes \mathbb{C} \xrightarrow{\sim} K_\mathbb{C}$ be the quasi-isomorphism given by the Poincaré lemma. Then $(K_\mathbb{Z}, (K_\mathbb{C}, F), \alpha)$ is a cohomological Hodge complex on X of weight 0. Its hypercohomology on X is isomorphic to the cohomology of X and carries a Hodge structure with Hodge filtration induced by F.

5 Mixed Hodge complexes

Definition 14. An *R-mixed Hodge complex* K consists of

- a complex K_R of R-modules, such that all $H^i(K_R)$ are finitely generated R-modules,
- a filtered complex $(K_{R\otimes\mathbb{Q}}, W)$ of $(R\otimes\mathbb{Q})$-vector spaces, and an isomorphism $K_R\otimes\mathbb{Q} \xrightarrow{\sim} K_{R\otimes\mathbb{Q}}$ in $D^+(R\otimes\mathbb{Q})$,
- a bi-filtered complex $(K_{\mathbb{C}}, W, F)$ of $\mathbb{C}$-vector spaces
- an isomorphism $\alpha : (K_{R\otimes\mathbb{Q}}, W)\otimes\mathbb{C} \xrightarrow{\sim} (K_{\mathbb{C}}, W)$ in $D^+W(\mathbb{C})$,

such that, for all n, the system $\mathrm{Gr}_n^W(K)$ consisting of

- the complex $\mathrm{Gr}_n^W(K_{R\otimes\mathbb{Q}})$ of $(R\otimes\mathbb{Q})$-vector spaces,
- the complex $\mathrm{Gr}_n^W(K_{\mathbb{C}}, F)$ of $\mathbb{C}$-vector spaces with induced filtration F,
- the isomorphism $\mathrm{Gr}_n^W(\alpha) : \mathrm{Gr}_n^W(K_{R\otimes\mathbb{Q}})\otimes\mathbb{C} \xrightarrow{\sim} \mathrm{Gr}_n^W(K_{\mathbb{C}})$,

forms an $(R\otimes\mathbb{Q})$-Hodge complex of weight n.

Definition 15. An *R-cohomological mixed Hodge complex* K on a topological space X consists of

- a complex K_R of sheaves of R-modules on X, such that all $H^i(X, K_R)$ are finitely generated R-modules,
- a filtered complex $(K_{R\otimes\mathbb{Q}}, W)$ of sheaves of $(R\otimes\mathbb{Q})$-vector spaces on X, and an isomorphism $K_R\otimes\mathbb{Q} \cong K_{R\otimes\mathbb{Q}}$ in $D^+(X, R\otimes\mathbb{Q})$,
- a bi-filtered complex $(K_{\mathbb{C}}, W, F)$ of sheaves of $\mathbb{C}$-vector spaces on X,
- an isomorphism $\alpha : (K_{R\otimes\mathbb{Q}}, W)\otimes\mathbb{C} \xrightarrow{\sim} (K_{\mathbb{C}}, W)$ in $D^+F(X, \mathbb{C})$.

such that, for all n, the system $\mathrm{Gr}_n^W(K)$ consisting of

- the complex $\mathrm{Gr}_n^W(K_{R\otimes\mathbb{Q}})$ of sheaves of $(R\otimes\mathbb{Q})$-vector spaces on X,
- the complex $(\mathrm{Gr}_n^W(K_{\mathbb{C}}), F)$ of sheaves of $\mathbb{C}$-vector spaces on X with induced filtration F,
- the isomorphism $\mathrm{Gr}_n^W(\alpha) : \mathrm{Gr}_n^W(K_{R\otimes\mathbb{Q}})\otimes\mathbb{C} \xrightarrow{\sim} \mathrm{Gr}_n^W(K_{\mathbb{C}})$,

is an $(R\otimes\mathbb{Q})$-cohomological Hodge complex on X of weight n.

Remark 16. If (K, W, F) is a (cohomological) MHC, then for all $m, p \in \mathbb{Z}$, also $(K[m], W[m-2p], F[p])$ is a (cohomological) MHC.

The following proposition shows how one obtains a mixed Hodge complex from a cohomological mixed Hodge complex.

Proposition 17. *Let $K = (K_R, (K_{R\otimes\mathbb{Q}}, W), (K_\mathbb{C}, W, F), \alpha)$ be an R-cohomological MHC. Then*

$$R\Gamma(K) := (R\Gamma(K_R), R\Gamma(K_{R\otimes\mathbb{Q}}, W), R\Gamma(K_\mathbb{C}, W, F), R\Gamma(\alpha))$$

is an R-MHC.

Proof. One needs to check that

$$\begin{aligned}\mathrm{Gr}_n^W(R\Gamma(K)) &= (\mathrm{Gr}_n^W(R\Gamma(K_{R\otimes\mathbb{Q}})), \mathrm{Gr}_n^W(R\Gamma(K_\mathbb{C}, F)), \mathrm{Gr}_n^W(R\Gamma(\alpha))) \\ &\cong (R\Gamma(\mathrm{Gr}_n^W(K_{R\otimes\mathbb{Q}})), R\Gamma(\mathrm{Gr}_n^W(K_\mathbb{C}, F)), R\Gamma(\mathrm{Gr}_n^W(\alpha)))\end{aligned}$$

is an $(R\otimes\mathbb{Q})$-Hodge complex. But this is just the image under $R\Gamma$ of the $(R\otimes\mathbb{Q})$-cohomological Hodge complex $\mathrm{Gr}_n^W(K)$ on X, which is indeed an $(R\otimes\mathbb{Q})$-Hodge complex by condition (CHC). □

6 MHS on the cohomology of a MHC

The goal of this section is to prove the following theorem.

Theorem 18 (Deligne)**.** *The cohomology of a mixed Hodge complex carries a mixed Hodge structure.*

The idea of the proof is to show that the terms ${}_WE_r^{p,q}$ of the weight spectral sequence of a MHC (K, W, F) are Hodge structures of weight q, and the differentials d_r are strictly compatible with the Hodge filtrations, that is, are morphisms of Hodge structures. Then d_r will vanish for $r \geq 2$, so the spectral sequence will degenerate at rank 2.

6.1 Spectral sequences of filtered complexes

Let $T : \mathcal{A} \to \mathcal{B}$ will be a left exact functor, and assume $\mathcal{A}$ has enough injectives.

Let (K, F) be an object of $D^+F\mathcal{A}$. The *spectral sequence defined by the filtered complex $RT(K, F)$* is [Hodge, 3.1.2.1] given by

$${}_FE_1^{p,q} = H^{p+q}(\mathrm{Gr}_F^p RT(K)) \Rightarrow \mathrm{Gr}_F^p R^{p+q}T(K).$$

Since $H^{p+q}(\mathrm{Gr}_F^p RT(K)) \cong H^{p+q}(RT(\mathrm{Gr}_F^p K)) = R^{p+q}T(\mathrm{Gr}_F^p K)$, we will write this sequence as

$${}_FE_1^{p,q} = R^{p+q}T(\mathrm{Gr}_F^p K) \Rightarrow \mathrm{Gr}_F^p(R^{p+q}T(K)).$$

This spectral sequence is called the *(hypercohomology) spectral sequence of the filtered complex K with respect to the functor T.*

Remark 19. Explicitly, the differentials d_1 of this spectral sequence are the image under T of $\mathrm{Gr}_F^p(K) \xrightarrow{\delta} (\mathrm{Gr}_F^{p+1} K)[1]$ given by the exact sequence

$$0 \to \mathrm{Gr}_F^{p+1}(K) \to F^pK/F^{p+2}K \to \mathrm{Gr}_F^p(K) \to 0.$$

6.2 Direct and recurrent filtrations

Let (K, W, F) be a bi-filtered complex of objects of an abelian category, bounded below. Similarly to the above, the filtration W yields the *weight spectral sequence*

$$ {}_W E_1^{p,q} = R^{p+q}(\mathrm{Gr}^W_{-p} K) \Rightarrow \mathrm{Gr}^W_{-p}(R^{p+q}T(K)). $$

The filtration F, assumed to be biregular, induces on the terms $E_r^{p,q}$ of the spectral sequence $E(K, W)$ the following filtrations.

Definition 20. The *first direct filtration* F_d on $E_r(K, W)$ is defined by

$$ F_d^p(E_r(K, W)) = \mathrm{im}\,(E_r(F^pK, W) \to E_r(K, W))\,. $$

The *second direct filtration* F_{d^*} on $E_r(K, W)$ is defined by

$$ F_{d^*}^p(E_r(K, W)) = \ker\,(E_r(K, W) \to E_r(K/F^pK, W))\,. $$

Definition 21. The *recurrent filtration* F_{rec} on $E_r^{p,q}$ is defined inductively as follows:

- On $E_0^{p,q}$, it is $F_{\mathrm{rec}} = F_d = F_{d^*}$.
- The recurrent filtration F_{rec} on $E_r^{p,q}$ induces a filtration on $\ker d_r$, which induces a recurrent filtration on F_{rec} on $E_{r+1}^{p,q}$ as a quotient of $\ker d_r$.

In general, these definitions will be different, but we need the following general properties.

Proposition 22. *(i) $F_d \subset F_{\mathrm{rec}} \subset F_{d^*}$ with equalities for $r = 0, 1$,*

(ii) the differential d_r is compatible with F_d and F_{d^*}.

Proof. *(i)* follows from [Hodge, Lemma 3.2.27]. *(ii)* follows from [Hodge, Proposition 3.2.29]. □

Remark 23. The point is that F_d and F_{d^*} interact well with respect to the differential d_r (but in general F_{rec} does not). However, the induced Hodge structure on the terms $E_{r+1}^{p,q}$ will be given by F_{rec}. Hence, these three different filtrations need to be compared, and it will turn out that, for ${}_W E_r^{p,q}$ coming from a MHC, the three filtrations will agree for all r!

6.3 Proof of Theorem 18

Let us start with the following lemma.

Lemma 24. *For all $r \geq 1$, the two direct filtrations and the recurrent filtration on ${}_W E_r^{p,q}$ agree (that is, $F_d = F_{\mathrm{rec}} = F_{d^*}$) and the differentials d_r are strictly compatible with the recurrent filtration $F = F_{\mathrm{rec}}$. For $r \geq 2$, they vanish.*

Proof. Proof by induction on r.

Suppose $r = 1$. By definition of a MHC, we have that $\mathrm{Gr}^W_{-p}(K)$ is a HC of weight $-p$. For $r = 1$, we have $F_d = F_{\mathrm{rec}} = F_{d^*}$ [Hodge, Lemma 3.2.27], so these filtrations equal the Hodge filtration on ${}_W E_1^{p,q} = H^{p+q}(\mathrm{Gr}^W_{-p} K)$. The differential d_1 is compatible with F_d and F_{d^*} [Hodge, Proposition 3.2.29] and hence with F_{rec}. Furthermore, d_1 commutes with complex conjugation since it is defined over $R \otimes \mathbb{Q} \subset \mathbb{R}$, so d_1 is also compatible with $\overline{F}_{\mathrm{rec}}$, that is, d_1 is strictly compatible with $F = F_{\mathrm{rec}}$.

The filtration F_{rec} is q-opposed to $\overline{F}_{\mathrm{rec}}$ (as ${}_W E_1^{p,q} = H^{p+q}(\mathrm{Gr}^W_{-p} K)$ is a HS of weight $-p+(p+q) = q$) and hence defines a HS of weight q on ${}_W E_2^{p,q} = \ker d_1^{p,q} / \operatorname{im} d_1^{p+1,q}$.

Now suppose $r \geq 2$ and assume that the statement holds for all $s < r$. Note that condition $(*r_0)$ of [Hodge, Theorem 3.2.30] is satisfied for $r_0 = r$, so $F_d = F_{\mathrm{rec}} = F_{d^*}$ also for rank r. This also shows that d_r is compatible with F_{rec}. Again, F_{rec} is q-opposed to $\overline{F}_{\mathrm{rec}}$. Now, $d_r : {}_W E_r^{p,q} \to {}_W E_r^{p+r,q-r+1}$ is a morphism of Hodge structures from weight q to weight $q - r + 1 < q$, so it must vanish. In particular, the weight spectral sequence degenerates at rank 2. ◻

Proposition 25. *Let K be an R-mixed Hodge complex.*

(i) The filtration $W[n]$ on $H^n(K_R) \otimes \mathbb{Q} \cong H^n(K_{R\otimes\mathbb{Q}})$, given by

$$W[n]_k H^n(K_{R\otimes\mathbb{Q}}) := \operatorname{im}\big(H^n(W_{k-n} K_{R\otimes\mathbb{Q}}) \to H^n(K_{R\otimes\mathbb{Q}})\big),$$

and the filtration F on $H^n(K_\mathbb{C}) \cong H^n(K_R) \otimes_R \mathbb{C}$, given by

$$F^p(H^n(K_\mathbb{C})) := \operatorname{im}\big(H^n(F^p K_\mathbb{C}) \to H^n(K_\mathbb{C})\big),$$

define an R-mixed Hodge structure

$$H^n(K) := (H^n(K_R), (H^n(K_{R\otimes\mathbb{Q}}), W), (H^n(K_\mathbb{C}), W, F)).$$

(ii) On the terms ${}_W E_r^{p,q}(K_\mathbb{C}, W)$, the recurrent filtration and the two direct filtrations coincide ($F_d = F_{\mathrm{rec}} = F_{d^*}$) and define the filtration F of a Hodge structure of weight q. Moreover, the differentials d_r are compatible with F.

(iii) The morphisms $d_1 : {}_W E_1^{p,q} \to {}_W E_1^{p+1,q}$ are strictly compatible with F.

(iv) The spectral sequence of $(K_{R\otimes\mathbb{Q}}, W)$ degenerates at rank 2. (Hence, ${}_W E_2^{p,q} = {}_W E_\infty^{p,q}$.)

(v) The spectral sequence of $(K_\mathbb{C}, F)$ degenerates at rank 1. (Hence, ${}_F E_1^{p,q} = {}_F E_\infty^{p,q}$.)

(vi) The spectral sequence of the complex $\mathrm{Gr}^p_F(K_\mathbb{C})$, with the induced filtration W, degenerates at rank 2.

Proof. *(ii)*, *(iii)* and *(iv)* follow from Lemma 24. For *(i)*, it must be shown that the graded pieces $\mathrm{Gr}^{W[n]}_k(H^n(K_{R\otimes\mathbb{Q}}))$, with the induced filtration F, are Hodge structures of weight k. But these graded pieces are precisely ${}_W E_\infty^{n-k,k}$, which we have seen to be Hodge structures of weight k. Furthermore, *(v)* has already been shown in [Hodge, Section 3.1], and follows from the fact that the differentials are strictly compatible with the filtration F. Finally, *(vi)* is just *(iv)* but switching F and W. ◻